

## CHAPTER 6

# RELATED RATE PROBLEMS



You should read this chapter if you need to review or you need to learn about

- Find related rates through implicit differentiation
- Solve distance-related rates
- Related rates for moving bodies
- Geometric-related rates
- Area-related rates
- Volume-related rates

Related rate problems relate one rate, written as a derivative, to another rate written as a derivative. An excellent example of a related rate problem, and one that is in nearly every calculus book including this one, is a ladder sliding down a wall. (See Fig. 6-1.) The top of the ladder is moving down the wall while the bottom of the ladder is moving away from the wall. The rate (speed) the top is moving down the wall can be related to the rate (speed) the bottom is moving away from the wall. This problem illustrates well the name of these problems; related rate problems.

A little review is in order. Related rate problems are similar to problems involving implicit differentiation. Equations in the form  $y = f(x)$  such as  $y = x^2 + 2x - 3$  are differentiated term by term according to the rules for differentiating polynomials, products, quotients, or whatever. Equations where the  $x$ 's and  $y$ 's are mixed together so the equation cannot be written as  $y = f(x)$  or  $x = f(y)$  (an  $x$  alone or a  $y$  alone on one side of the equation) are differentiated implicitly.

For example, the equation  $2xy^2 + xy^3 = 0$  must be differentiated implicitly as

$$2y^2 dx + 4xy dy + 3xy^2 dy + y^3 dx = 0$$

with  $dy/dx$  formed by grouping and rearranging.

If  $x$  and  $y$  could both change over time then a related rate associated differentiation of this equation would be

$$2y^2 \frac{dx}{dt} + 4xy \frac{dy}{dt} + 3xy^2 \frac{dy}{dt} + y^3 \frac{dx}{dt} = 0$$

In this statement,  $\frac{dx}{dt}$  is directly related to  $\frac{dy}{dt}$

$$(2y^2 + y^3) \frac{dx}{dt} = -(4xy + 3xy^2) \frac{dy}{dt} \quad \text{or} \quad \frac{dx}{dt} = -\frac{4xy + 3xy^2}{2y^2 + y^3} \frac{dy}{dt}$$

This is an example of a related rate differentiation. Now take a look at perhaps not the simplest related rate problem, but possibly the simplest to visualize. Notice how this problem is written. The general situation is described, then a rate is specified and the related rate is requested for a certain condition.

**Example 6-1** A 7-m-long ladder is sliding down a wall. The bottom of the ladder is pulled from the wall at 1.5 m/s. What is the rate at which the top of the ladder is going down when the bottom is 3 m from the wall?

**Solution:** Help to visualize the problem by sketching a ladder leaning against a wall with the bottom being pulled out from the wall at  $dx/dt = 1.5$  m/s (Fig. 6-1).

The question, written in mathematical language, is: “What is  $dy/dt$  when  $x = 3$  m and  $dx/dt = 1.5$  m/s?”

In max-min problems the defining equation is a mathematical statement of the problem. In related rate problems the defining equation is sometimes a little more obscure, actually sometimes a lot more obscure! Look at the ladder in the graphic and think of a way to relate  $x$  to  $y$ . Don't start by trying to write the  $dx/dt$  and  $dy/dt$ . The rates come out of the differentiation.

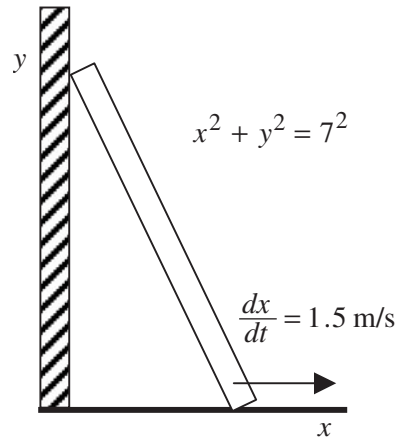


Fig. 6-1

**Quick Tip**



The hardest part of related rate problems is to see, and then write down, a relationship between the variables. Writing this defining equation that ties the variables together is the key step in related rate problems.

In this problem the Pythagorean theorem for a right triangle relates  $x$  and  $y$ .

The defining equation is  $x^2 + y^2 = 7^2$  and taking  $\frac{d}{dt}$  we write

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

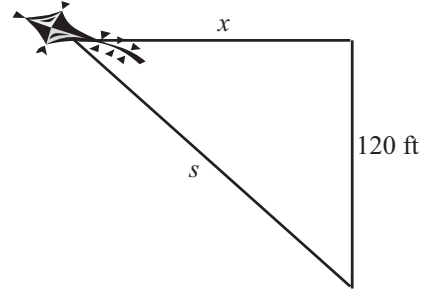
Now the numbers can be put in the equation to find  $dy/dt$  when  $dx/dt = 1.5$  m/s and  $x = 3$  m. What about the denominator? The  $y$  can be determined from the Pythagorean relation  $y = \sqrt{7^2 - 3^2} \approx 6.3$ . With these numbers,  $dy/dt$  is calculated as

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{3}{6.3} (1.5 \text{ m/s}) = -0.71 \text{ m/s}$$

The top of the ladder is coming down the wall at 0.71 m/s when the bottom of the ladder is 3 m away from the wall and moving at 1.5 m/s.

**Example 6-2** A girl is flying a kite. The kite is moving horizontally at a height of 120 ft when 250 ft of string is out and the rate of increase in string length is 2 ft/s. How fast is the kite moving in the  $x$ -direction for these conditions?

**Solution:** Visualize the problem and set up a right triangle with the height, horizontal direction, and string. In this problem the kite only moves horizontally, and the string is straight—idealized—but the conditions make for a problem that can be solved.



**Fig. 6-2**

Referring to Fig. 6-2, the problem question can be written in mathematical terms as:

What is  $\frac{dx}{dt}$ , when  $\frac{ds}{dt} = 2$ , the height of the kite is 120 ft, and the distance out is 250 ft?

Go back over the problem statement and practice changing the problem statement into this mathematical statement. One of the more challenging parts of any calculus problem is translating the words into mathematical statements.

The Pythagorean theorem relates the variables  $x$  and  $s$  in the right triangle:

$$120^2 + x^2 = s^2.$$

Take  $\frac{d}{dt}$  to get  $2x \frac{dx}{dt} = 2s \frac{ds}{dt}$  or  $\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$ . The  $ds/dt$  rate (2 ft/s) is given in the problem as is the height (120 ft) and the distance out (250 ft). The  $x$  value for these conditions can be calculated from the Pythagorean theorem:

$$x^2 = s^2 - 120^2 \quad \text{or} \quad x = \sqrt{250^2 - 120^2} = 219$$

Now the numbers can be put into the formula for  $dx/dt$ :

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt} = \frac{250 \text{ ft}}{219 \text{ ft}} (2 \text{ ft/s}) = 2.3 \text{ ft/s}$$

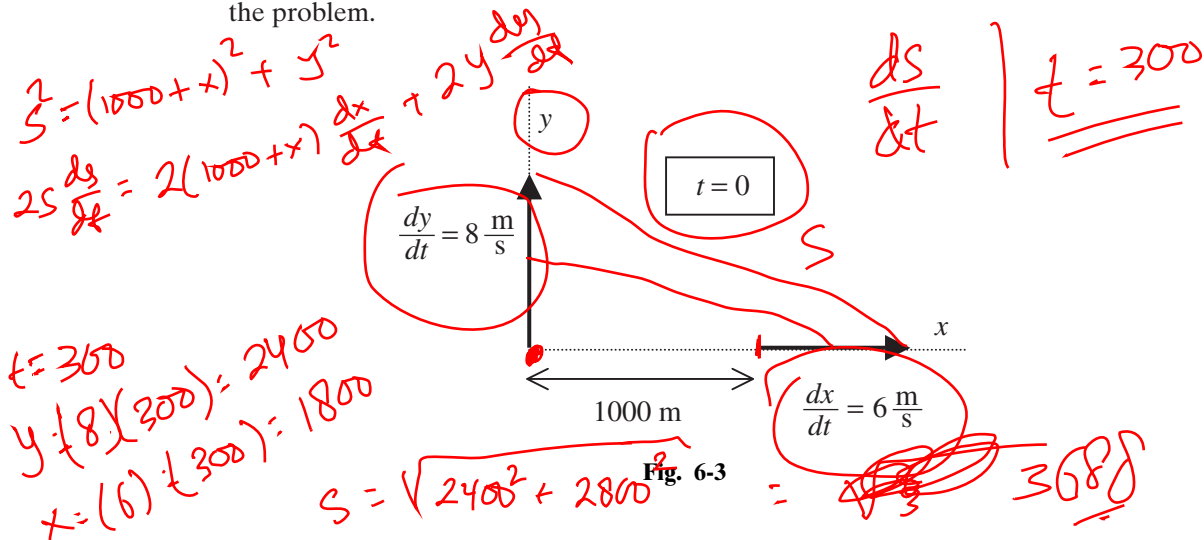
When the kite is 250 ft away from the girl, at a height of 120 ft, and the string is going out at 2 ft/s, the kite is moving 2.3 ft/s horizontally.

These first two problems have utilized the Pythagorean theorem as their defining or “getting started” equation. Related rate problems use a variety of defining statements to tie the variables together. As you go through this chapter be aware of the various techniques for relating the variables. If you see a related rate problem on a test that can be analyzed with the Pythagorean theorem, you will know how to do that problem.

This next problem uses the Pythagorean theorem but it has another little twist. The information for the problem is given primarily in terms of rates, and the solution involves three different rates.

**Example 6-3** Two ships are traveling at right angles. The first ship, traveling at 8 m/s, crosses the path of the second ship when it is 1000 m away (from the point where the paths cross) and traveling at 6 m/s. What are their positions, separation, and rate of separation 300 s after their paths cross?

**Solution:** Diagram the problem on an  $x$ - $y$  coordinate system with the first ship going in the  $y$ -direction and the second ship going in the  $x$ -direction. Figure 6-3 is for  $t = 0$ , the time when the ships cross paths. The drawing helps to visualize the problem.



The position of the first ship at any time  $t$  is  $y = (8 \text{ m/s})t$ . The position of the second ship at any time  $t$  is  $x = 1000 \text{ m} + (6 \text{ m/s})t$ . The separation of the ships is from the Pythagorean theorem  $s = \sqrt{x^2 + y^2}$ .

The position of the first ship at 300s is its speed (8m/s) times the 300s

$$y|_{300} = (8 \text{ m/s})(300 \text{ s}) = 2400 \text{ m}$$

The position of the second ship at 300 s is the 1000 m plus the 6 m/s times the 300 s

$$x|_{300} = 1000 \text{ m} + (6 \text{ m/s})(300 \text{ s}) = (1000 + 1800) \text{ m} = \underline{2800 \text{ m}}$$

The separation of the ships is a straight Pythagorean theorem problem.

$$s = \sqrt{2400^2 + 2800^2} = \underline{3688 \text{ m}}$$

The rate at which they are separating is the fun, that is to say calculus, part of the problem. The rate at which they are separating is, in calculus talk,  $ds/dt$ , and we already have the  $dx/dt$  and  $dy/dt$ . Start with the separation written in Pythagorean theorem form  $s = (x^2 + y^2)^{1/2}$  and differentiate, carefully.

Writing

$$2s \frac{ds}{dt} = 2(1000+x) \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$ds = \frac{1}{2}(x^2 + y^2)^{-1/2} d(x^2 + y^2)$$

as the first step will help to prevent errors with (1/2)s and the minus signs.

Continuing,  $ds = \frac{1}{2}(x^2 + y^2)^{-1/2} (2xdx + 2ydy)$  and finally

$$\frac{ds}{dt} = \frac{1}{(x^2 + y^2)^{1/2}} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right)$$

$\downarrow \cdot 3688 \cdot \frac{ds}{dt} = 2(2800)6 + 2(2400)(8)$

This rate of separation is to be evaluated at  $t = 300$  s (Fig. 6-4).

$$\left. \frac{ds}{dt} \right|_{300} = \frac{1}{3688 \text{ m}} [(2800 \text{ m})(6 \text{ m/s}) + (2400 \text{ m})(8 \text{ m/s})] = 9.8 \frac{\text{m}}{\text{s}}$$

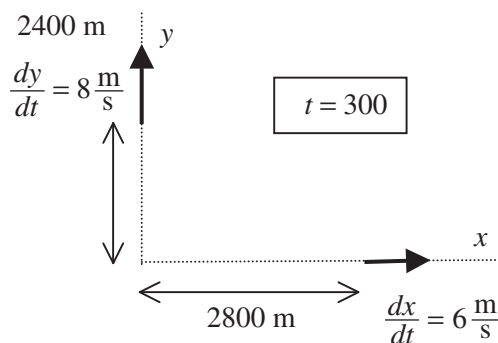


Fig. 6-4

**Remember**

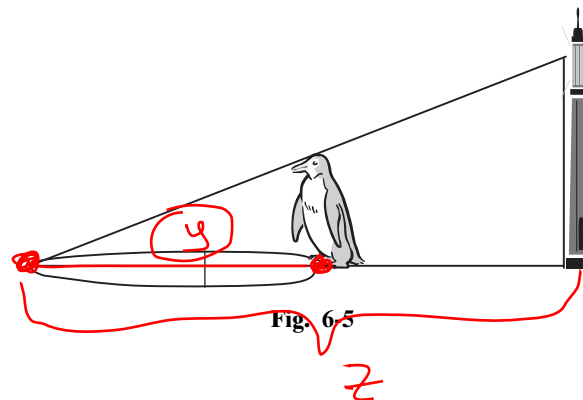


These next two problems utilize similar triangles to write the defining equation for the problem. The first problem, concerning the rate a shadow of something is moving, is in nearly every calculus book. The following problem concerning a conical-shaped container is also in nearly every calculus book in one form or another. If you know how to use similar triangles to “get started” on a problem you will have mastered yet another category of related rate problems.

**Example 6-4** A 3-ft tall penguin (Penny) is taking a leisurely stroll at 0.5 ft/s away from a 12-ft tall penguin way light. What is the length of her shadow and how fast is the tip of her shadow moving when she is 40 ft away from the light?

**Solution:** When you see a triangle in a related rate problem look for similar triangles. Don’t start the problem looking for derivatives. Concentrate on the defining equation for the problem. The derivatives come later.

Your first order of business in a related rate problem is to find relationships between the variables. In this problem set up the triangle, complete with known numbers, and then label some of the distances. The change in length of the hypotenuse of this triangle is not what we are looking for. It is lengths along the ground: the length from the light to Penny and the length of her shadow. Take  $x$  as the length from beneath the light to Penny, and  $z$  as the length from beneath the light to the end of her shadow. The length of her shadow is  $z - x$ . Draw this triangle (refer to Figs. 6-5 and 6-6).



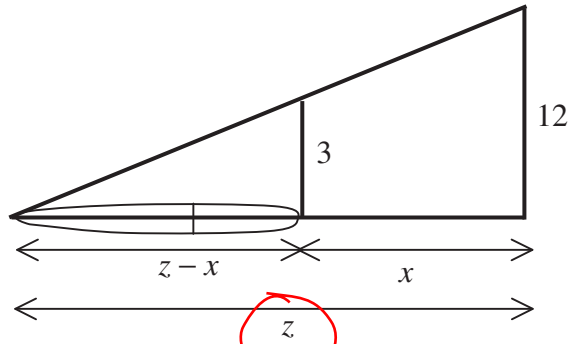


Fig. 6-6

Notice that the triangle with sides  $z - x$  and 3 is similar to the triangle with sides  $z$  and 12. Similar triangles are triangles with the same angles and their sides in proportion. This means that the ratios of the sides are equal.

$$\frac{z - x}{3} = \frac{z}{12}$$

Eliminating the fraction,  $12z - 12x = 3z$  or  $9z = 12x$  or  $3z = 4x$ , produces a simple relationship between  $x$  and  $z$ . The related derivative rates are

$$\frac{dz}{dt} = \frac{4}{3} \frac{dx}{dt}$$

Notice that  $x$  and  $z$  don't enter into the rate relationship. Penny is moving at

$$\frac{dx}{dt} = 0.5 \frac{\text{ft}}{\text{s}}$$

so the tip of her shadow is moving at

$$\frac{dz}{dt} = \frac{4}{3} \frac{dx}{dt} = \frac{4}{3} \left( 0.5 \frac{\text{ft}}{\text{s}} \right) = 0.67 \frac{\text{ft}}{\text{s}}$$

Since Penny is walking away from the light at 0.5 ft/s and the tip of her shadow is growing at 0.67 ft/s her shadow is getting longer as she moves away from the light.

As an exercise, go to a desk or table with a lamp. Place a pencil near the lamp and observe the length of the shadow. The pencil should be a foot or so from the light and perhaps slightly tilted. Now move the pencil away from the light and observe the shadow. The shadow will grow and the tip of the shadow will move faster than the pencil.

**Example 6-5** A conical container of base radius 5 ft and height 10 ft is being filled with sand at the rate of  $2\text{ ft}^3/\text{min}$ . How fast is the level of the sand rising when it is 6 ft above the apex of the conical container? (See Fig. 6-7.)

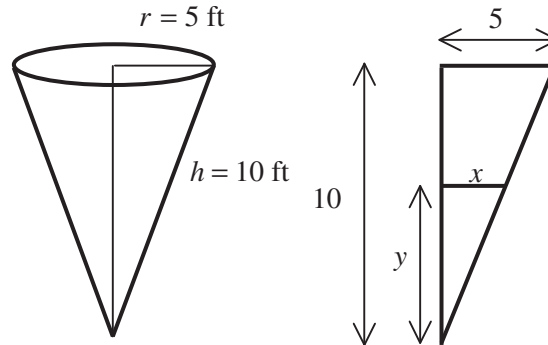


Fig. 6-7

**Solution:** The formula for the volume of a cone is, from the Mathematical Tables in the back of the book,  $V = (1/3)\pi r^2 h$ .

The dimensions defining the cone are given in the problem so calculating the total volume of the container is not a problem.

Sketch the cone, and next to the cone sketch the profile of the entire cone and a partially filled cone with radius  $x$  and height  $y$ . This is another similar triangles problem! The radius to height ratio is the same for any radius and depth. In this case the similar triangles are the ones with sides  $x$  and  $y$ , and 5 and 10.

The similar triangle statement is  $\frac{5}{10} = \frac{x}{y}$  or  $x = \frac{y}{2}$ . The question “How fast is the level of the sand rising . . .?” means, what is  $dy/dt$ ? Knowing  $dV/dt$  and requiring  $dy/dt$ , we need to write  $V$  in terms of  $y$  only. Time derivatives of  $V$  in terms of  $y$  will produce a relation between  $dV/dt$  and  $dy/dt$ .

Substitute in the  $V$  equation:

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3.$$

And taking derivatives produces

$$\frac{dV}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt} \quad \text{or} \quad \frac{dy}{dt} = \frac{4}{\pi y^2} \frac{dV}{dt}.$$

Adding numbers for  $y = 6$  ft,

$$\frac{dy}{dt} = \frac{4}{\pi y^2} \frac{dV}{dt} = \frac{4}{(36 \text{ ft}^2)\pi} \left( 2 \frac{\text{ft}^3}{\text{min}} \right) = 0.071 \frac{\text{ft}}{\text{min}}.$$

At a depth of 6 ft the sand is rising at 0.071 ft/min.

Another category of related rate problems involves increasing or decreasing area, volume, radius of a sphere, or some other geometric property. These next two problems involve geometry. In general, geometry problems are not overly difficult, usually involving just one equation.

**Example 6-6** A circular oil slick is forming in such a way that the radius of the slick is increasing at a constant rate of 12 ft/hr. What will be the rate of area increase when the slick has radius 300 ft? (See Fig. 6-8.)

**Solution:** The area is related to the radius by  $A = \pi r^2$  (see the Mathematical Tables). The rate of  $A$  and the rate of  $r$  are directly available from this one equation.

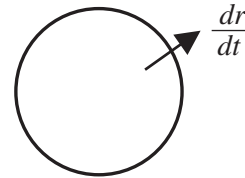


Fig. 6-8

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Using the numbers given in the problem

$$\left. \frac{dA}{dt} \right|_{300} = 2\pi r \frac{dr}{dt} = 2\pi (300 \text{ ft}) \left( 12 \frac{\text{ft}}{\text{hr}} \right) = 22,600 \frac{\text{ft}^2}{\text{hr}}$$

The area of the oil slick is increasing at 22,600 ft<sup>2</sup>/hr when the radius is 300 ft.

**Example 6-7** The deployment of the safety air bags in automobiles is very much like blowing up a balloon. If the air bag expands under a volume change of 10,000 cm<sup>3</sup>/s, what is the radial change when the bag has 5 cm radius and 10 cm radius? What is the significance of your result?

**Solution:** The volume of the bag is

$$V = \frac{4}{3} \pi r^3$$

The rate relations are

$$\frac{dV}{dt} = \frac{4}{3}\pi \left( 3r^2 \frac{dr}{dt} \right) \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

Now put in the numbers for the 5 cm and 10 cm radii.

$$\left. \frac{dr}{dt} \right|_5 = \frac{1}{4\pi(5 \text{ cm})^2} 10,000 \frac{\text{cm}^3}{\text{s}} = 32 \frac{\text{cm}}{\text{s}}$$

$$\left. \frac{dr}{dt} \right|_{10} = \frac{1}{4\pi(10 \text{ cm})^2} 10,000 \frac{\text{cm}^3}{\text{s}} = 8 \frac{\text{cm}}{\text{s}}$$

Doubling your distance from the air bag decreases its speed when it hits you by one-fourth. Conclusion: Sit as far away from the air bag as you can!

**Example 6-8** An obstruction in an artery is to be removed by inflating a spherical balloon in the artery. The rate of increase of the radius of the balloon must be limited to 1 mm/min when the radius is 4 mm. What is the maximum volume rate increase, the rate at which oxygen is pumped into the balloon, corresponding to this radius rate increase? (See Fig. 6-9.)

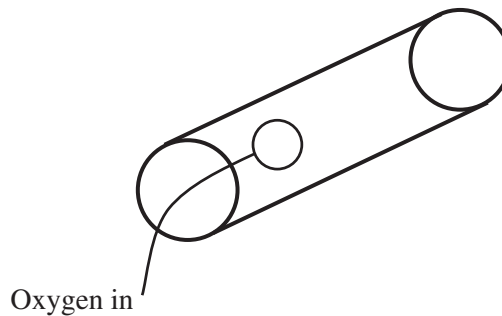


Fig. 6-9

**Solution:** The volume of a sphere is  $V = (4/3)\pi r^3$  (see the Mathematical Tables).

Again, the rate relations are immediately available from this equation for the volume of a sphere.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Evaluating at  $r = 4$  mm and  $\frac{dr}{dt} = 1 \frac{\text{mm}}{\text{min}}$

$$\left. \frac{dV}{dt} \right|_4 = 4\pi(4\text{mm})^2 \left( 1 \frac{\text{mm}}{\text{min}} \right) = 201 \frac{\text{mm}^3}{\text{min}}$$

The maximum rate that the balloon can be filled at the 4 mm radius is  $201 \text{ mm}^3/\text{min}$ .

These last two problems are illustrative of problems where the formulas are given to you. In most of these types of problems, differentiating the formula is the challenge.

**Example 6-9** When the price of a certain product is  $p$  dollars per unit, customer demand is  $x$  hundreds of units (per month). The relation between  $p$  and  $x$  is  $x^2 + 2px + 0.5p^2 = 80$ . When the price is \$4.00 and dropping at the rate of \$0.25 per month, what is the rate of increase in demand?

**Solution:** This equation requires an implicit type of differentiation to find  $dp/dt$ , the rate of price change, and  $dx/dt$ , the rate of demand change.

$$2x \frac{dx}{dt} + 2p \frac{dx}{dt} + 2x \frac{dp}{dt} + p \frac{dp}{dt} = 0 \quad \text{or} \quad (2x + 2p) \frac{dx}{dt} + (2x + p) \frac{dp}{dt} = 0$$

$$\text{or} \quad \frac{dx}{dt} = - \frac{2x + p}{2x + 2p} \frac{dp}{dt}$$

The rate of price change,  $dp/dt$ , is given in the problem as is  $p$ , the price. The demand rate,  $x$ , is not given and must be computed from the original equation. Substituting for  $p = 4$  ( $p = \$4.00$ ) in  $x^2 + 2px + 0.5p^2 = 80$  yields  $x^2 + 8x + 8 = 80$  or  $x^2 + 8x - 72 = 0$ . The quadratic formula produces two answers. The positive 5.4 is the realistic one.

$$x = \frac{-8 \pm \sqrt{64 - 4(1)(-72)}}{2(1)} = \frac{-8 \pm 18.8}{2} = 5.4, -14.4$$

With all the needed values,  $dx/dt$  can be evaluated. Watch the signs closely.

$$\left. \frac{dx}{dt} \right|_4 = - \frac{2(5.4) + 4}{2(5.4) + 8} (-0.25) = \frac{14.8}{18.8} (0.25) = 0.20$$

The demand rate is increasing by 0.20 hundreds of units per month when the price is \$4.00 and dropping at the rate of \$0.25 per month. Carrying the units through this problem is difficult because the constants in the original equation must have the appropriate units to make each term in the equation have the same units.

**Example 6-10** The amount of trash, measured in thousands of pounds, accumulating in a city dump follows the formula  $T = 1.3p^2 - 100p + 30$ , where  $p$  is the population in hundreds of thousands. What is the rate of trash increase when the population is 200 thousand and increasing by 0.2 thousand (0.1%) per month?

**Solution:** Relating the rate of trash increase,  $dT/dt$ , to the population increase,  $dp/dt$ , comes directly from implicit type differentiation of the expression for the amount of trash.

$$\frac{dT}{dt} = 2.6p \frac{dp}{dt} - 100 \frac{dp}{dt} = (2.6p - 100) \frac{dp}{dt}$$

The population and the rate of increase in population are given in the problem so we have

$$\begin{aligned} \left. \frac{dT}{dt} \right|_{200} &= [(2.6)(200) - 100] [0.2 \text{ thousand per month}] \\ &= 84 \text{ thousand of pounds per month.} \end{aligned}$$

This is also an interesting max-min problem. Take  $\frac{dT}{dp} = (2.6p - 100)$  and set

equal to zero to find  $p = 38$ . The second derivative  $\frac{d^2T}{dp^2} = 2.6$  so the point  $p = 38$  is a minimum.

The minimum in trash accumulation rate is at 38 thousand people, but at 200 thousand the curve becomes progressively more positive and the trash problem progressively worse.

**Example 6-11** The number of CD players that can be manufactured by a certain company follows a Cobb-Douglas type of production model, where  $q$ , the production output, depends on the number of workers,  $n$ , and the number of automatic assembly machines,  $r$ , according to  $q = 20n^{0.6}r^{0.4}$ . With 20 workers the company is producing 300 CDs per day and has sufficient revenue to purchase one automatic assembly machine per month. How many workers per month should be laid off to maintain constant production?

**Solution:** The first step in the problem is to perform an implicit differentiation remembering that  $q$  has to remain constant.

$$0 = 20[n^{0.6}(0.4)r^{-0.6}dr + r^{0.4}(0.6)n^{-0.4}dn]$$

Now write the rate relationship. This could have been written directly.

$$0.4\left(\frac{n}{r}\right)^{0.6}\frac{dr}{dt} = -0.6\left(\frac{r}{n}\right)^{0.4}\frac{dn}{dt}$$

$$\frac{dn}{dt} = -\frac{2}{3}\left(\frac{n}{r}\right)\frac{dr}{dt}$$

It takes 20 workers to produce 300 CDs so the number of automatic assembly machines can be calculated from the original equation.

$$q = 20n^{0.6}r^{0.4} \Rightarrow 300 = (20)(20)^{0.6}r^{0.4}$$

$$r^{0.4} = \frac{300}{(20)(20)^{0.6}} = \frac{15}{20^{0.6}}$$

$$[r^{0.4}]^{2.5} = r = \left[\frac{15}{(20)^{0.6}}\right]^{2.5} = \frac{(15)^{2.5}}{(20)^{1.5}} = 9.7$$

Now calculate the related rate.

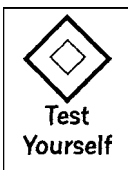
$$\frac{dn}{dt} = -\frac{2}{3}\left(\frac{20}{9.7}\right)(1) = -1.4$$

Adding one automatic assembly machine and releasing 1.4 workers per month will maintain production.



It's a Wrap

- ✓ Relate the variables in the problem
- ✓ Use a diagram if that will help
- ✓ Use a variation of implicit differentiation to write the rates
- ✓ Calculate specific variable values from the defining equation



Test Yourself

### PROBLEMS

1. For the equation  $x^2 + y^2 = 20$  find  $dy/dt$  when  $x = 4$  and  $dx/dt = -6$ .
2. For  $x^2 - 2xy + y^2 = 0$  find  $dx/dt$  when  $y = 2$  and  $dy/dt = 3$ .

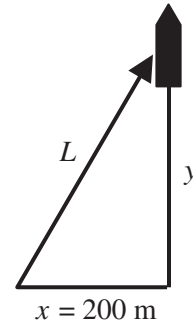
3. A rocket launched vertically is tracked by a camera 200 m horizontally away from the launch point. What is the vertical speed of the rocket when it is 600 m from the camera with this distance (from the camera) increasing at 80 m/s?
4. The cost in dollars per day to produce thumb drives follows  $C(x) = 4000 - 12x + \frac{x^2}{200}$  where  $x$  is the number of thumb drives produced each day. The present production level is 200 thumb drives per day and this rate is increasing at 2 per day. How fast is the average cost changing? Remember that average cost is  $C(x)$  over  $x$ .
5. At a price,  $p$ , a manufacturer is willing to supply  $x$  number of products according to the equation  $x^2 - xp - p^2 = 30$ . What is the rate of supply of  $x$  when  $p$  is \$9.00 and increasing at \$0.20 per week?
6. The mass of a raindrop is  $M = \rho \frac{4}{3} \pi r^3$  where  $\rho$  is the density of water, 1 g/cm<sup>3</sup>, and  $r$  is the radius of the drop. At what rate is the radius of a raindrop changing when it has a radius of 10 mm and is gaining water at the rate of 0.01 g/min?
7. An observer with a telescope is observing the approach of an airplane traveling 700 km/hr towards a point directly over the telescope. The airplane remains at a constant height of 9.0 km. What is the angular rate of change (in rad/sec) for the telescope when the airplane is 24 km horizontally from the telescope?
8. An adiabatic process follows the pressure-volume formula  $PV^{1.4} = C$ , a constant. For  $P = 2$  N/m<sup>2</sup>,  $V = 0.8$  m<sup>3</sup> and  $V$  is increasing at 0.6 m<sup>3</sup>/s, what is happening to  $P$ .
9. The period, the time for one back-and-forth excursion, of a pendulum is  $T = 2\pi\sqrt{\frac{L}{g}}$ , where  $L$  is the length of the pendulum and  $g$  is the acceleration due to gravity equal to 9.8 m/s<sup>2</sup>. If the length of the pendulum stretches by 0.01% over 1 year, what is the change in the period?
10. The Poiseuille's law for flow through a cylindrical pipe (blood vessel) is:  $v = \frac{K}{L}(R^2 - r^2)$ , where  $v$  is the velocity,  $K$  is a constant equal to 6 1/s,  $L$  is the length of the pipe,  $R$  is the radius of the pipe and  $r$  is the distance out from the center line of the pipe. For a 100-cm pipe of radius 0.2 cm, find the rate that the flow velocity is changing half-way between the center line and the wall when the pipe is contracting at a rate of 0.0004 cm/s?

**ANSWERS**

1. Start with the implicit derivative  $2xdx + 2ydy = 0$  and proceed to time derivatives  $y\frac{dy}{dt} = -x\frac{dx}{dt}$  and to  $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$ . Now stop and calculate  $y$  for  $x = 4$ .  $4^2 + y^2 = 20 \Rightarrow y = 2 \Rightarrow \frac{dy}{dt} = -\frac{4}{2}(-6) = 12$ .

2.  $2x \frac{dx}{dt} - 2x \frac{dy}{dt} - 2y \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow (x - y) \frac{dx}{dt} = (x - y) \frac{dy}{dt}$  Whatever the values for  $x$  and  $y$  are,  $\frac{dx}{dt} = \frac{dy}{dt}$ , so  $\frac{dx}{dt} = 3$ .

3. Use a diagram. This is a Pythagorean theorem problem where the variables are from the statement:  $x^2 + y^2 = L^2$ . In this problem the  $x$  is constant so take the rate derivatives.  $2y \frac{dy}{dt} = 2L \frac{dL}{dt} \Rightarrow \frac{dy}{dt} = \frac{L}{y} \frac{dL}{dt}$  Calculate  $y$  when  $L$  is 600 m.  $200^2 + y^2 = 600^2 \Rightarrow y = 566$  Now calculate the vertical speed of the rocket.



$$\frac{dy}{dt} = \frac{600}{566} 80 \text{ m/s} = 85 \text{ m/s}$$

4. The average cost is  $\overline{C(x)} = \frac{C(x)}{x} = 4000x^{-1} - 12 + \frac{x}{100}$ . The rate statement is  $\frac{d\overline{C(x)}}{dt} = (4000)(-1)x^{-2} \frac{dx}{dt} + \frac{1}{100} \frac{dx}{dt} = \left[ \frac{1}{100} - \frac{4000}{x^2} \right] \frac{dx}{dt} = \left[ \frac{1}{100} - \frac{4000}{200^2} \right] (2) = -0.18$

At this point the average cost of producing the thumb drives is dropping at the rate of \$0.18 per day.

5. Write the rate statement:  $2x \frac{dx}{dt} - x \frac{dp}{dt} - p \frac{dx}{dt} - 2p \frac{dp}{dt} = 0$ . Want to know  $\frac{dx}{dt}$  when  $p = 9$  and  $\frac{dp}{dt} = 0.20$  per week.

$$(2x - p) \frac{dx}{dt} = (2p + x) \frac{dp}{dt} \Rightarrow \frac{dx}{dt} = \frac{2p + x}{2x - p} \frac{dp}{dt}$$

Now find  $x$  when  $p = 9$ .  $x^2 - 9x - 81 = 30$  This equation is solved by quadratic formula to yield  $x = 16$ , so

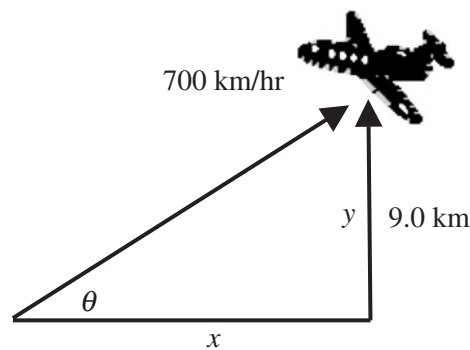
$$\frac{dx}{dt} = \frac{18 + 16}{32 - 9} \frac{dp}{dt} = 1.5(0.20) = 0.30$$

The manufacturer is willing to supply 0.30 more items per week or roughly 1 more every 3 weeks for these conditions.

6.  $\frac{dM}{dt} = \rho \frac{4}{3} \pi (3)r^2 \frac{dr}{dt} = 4\rho\pi r^2 \frac{dr}{dt}$   
 $\frac{dr}{dt} = \frac{1}{4(1 \text{ g/cm}^3)\pi(1 \text{ cm})^2} \left( 0.01 \frac{\text{g}}{\text{min}} \right) = 0.0025 \frac{\text{cm}}{\text{min}}$

7. This requires a drawing. Referring to the diagram,  $\frac{dx}{dt} = -700$  km/hr and  $y = 9.0$  km. To relate the angle use the tangent function,  $\tan \theta = \frac{y}{x}$  or  $9 = x \tan \theta$ . Take the rate derivatives.

$$x \sec^2 \theta \frac{d\theta}{dt} + \tan \theta \frac{dx}{dt} = 0$$



When  $x$  is 24, the angle is  $21^\circ$ , so  $24(\sec^2 21^\circ) \frac{d\theta}{dt} = -\tan 21^\circ \left(-700 \frac{\text{km}}{\text{hr}}\right)$

$$\frac{d\theta}{dt} = \frac{(\cos^2 21^\circ) \tan 21^\circ}{24 \text{ km}} (700) \frac{\text{km}}{\text{hr}} = 9.8 \frac{1}{\text{hr}} \frac{\text{hr}}{60 \text{ min}} = 0.16 \frac{\text{rad}}{\text{min}}$$

This is roughly  $0.3^\circ$  per min.

8. Take the rate derivatives:

$$P(1.4)V^{0.4} \frac{dV}{dt} + V^{1.4} \frac{dP}{dt} = 0 \quad \frac{dP}{dt} = -\frac{(1.4)PV^{0.4}}{V^{1.4}} \frac{dV}{dt} = -\frac{(1.4)P}{V} \frac{dV}{dt}$$

Put in the numbers.

$$\frac{dP}{dt} = -\frac{(1.4)(2 \text{ N/m}^2)}{(0.8 \text{ m}^3)} [0.6 \text{ m}^3/\text{s}] = 2.1 \frac{\text{N/m}^2}{\text{s}}$$

9. Start with  $T = \frac{2\pi}{\sqrt{g}} L^{1/2}$  and take the rate derivative.  $\frac{dT}{dt} = \frac{2\pi}{\sqrt{g}} \frac{1}{2} L^{-1/2} \frac{dL}{dt}$ .

The  $\frac{dL}{dt} = 0.0001 L$ . Put this into the rate equation.  $\frac{dT}{dt} =$

$$\frac{1}{2} \frac{2\pi}{\sqrt{g}} \frac{1}{\sqrt{L}} (0.0001)L = \frac{0.0001}{2} 2\pi \sqrt{\frac{L}{g}}. \text{ Notice that this can be written in}$$

terms of the original period.  $\frac{dT}{dt} = 0.00005 T$ . Writing this another way is instructive.  $\frac{dT}{T} = 0.00005 dt$ . This states that the fractional change in the period is 0.00005 per year.

10. Write the general rate statement.  $\frac{dv}{dt} = \frac{K}{L}(-)2r\frac{dr}{dt}$  Now put in the numbers.

$$\frac{dv}{dt} = -2\frac{6/s}{1.0 \text{ m}} (0.002 \text{ m})\left(0.0004 \frac{\text{cm}}{\text{s}}\right) = -1.0 \times 10^{-5} \frac{\text{cm}}{\text{s}^2}.$$